Metoda per partes

1. $\int xe^x dx =$

$$u' = e^x,\ v = x,\ v' = 1, u = \int e^x dx = e^x,$$

$$= xe^x - \int e^x \cdot 1 dx = xe^x - e^x = e^x(x-1) + C.$$

2. $\int x^2 e^x dx =$

$$u' = e^x,\ v = x^2, u = e^x,\ v' = 2x.$$

$$= x^2 e^x - 2\int e^x \cdot x dx = x^2 e^x - 2\left(e^x(x-1)\right) = e^x(x^2 - 2x + 2) + C.$$

3. $\int \frac{x^2}{e^x} dx =$

$$u' = \frac{1}{e^x},\ v = x^2,\ v' = 2x, u = \int \frac{1}{e^x} dx = -\frac{1}{e^x}.$$

$$= -\frac{1}{e^x}x^2 - \int -\frac{1}{e^x} \cdot 2x dx = -\frac{x^2}{e^x} + 2\int \frac{x}{e^x} dx =$$

$$u' = \frac{1}{e^x},\ v = x,\ v' = 1, u = -\frac{1}{e^x}.$$

$$= -\frac{x^2}{e^x} + 2\left(-\frac{1}{e^x} \cdot x - \int -\frac{1}{e^x} dx\right) = -\frac{x^2 + 2x + 2}{e^x} + C.$$

4. $\int \ln x dx =$

$$u' = 1,\ v = \ln x,\ v' = (\ln x)' = \frac{1}{x}, u = \int dx = x.$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x\ln x - x + C = x(\ln x - 1) + C = x \ln \frac{x}{e} + C, \text{pro } x > 0.$$

5. $\int \frac{\ln x}{x} dx =$

$$u' = \frac{1}{x}, \;\; v = \ln x, \;\; v' = \frac{1}{x}, u = \ln x.$$

$$= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x} dx = \ln^2 x - \int \frac{\ln x}{x} dx$$

$$\int \frac{\ln x}{x} dx = \ln^2 x - \int \frac{\ln x}{x} dx$$

$$2 \int \frac{\ln x}{x} dx = \ln^2 x$$

$$\int \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 x + C, \text{pro } x > 0.$$

6. $\int \frac{\arctan x}{1+x^2} dx =$

$$u' = \frac{1}{1+x^2}, \;\; v = \arctan x, \;\; v' = \frac{1}{1+x^2}, u = \int \frac{1}{1+x^2} dx = \arctan x.$$

$$= \arctan x \cdot \arctan x - \int \arctan x \frac{1}{1+x^2} dx = \arctan^2 x - \int \frac{\arctan x}{1+x^2} dx$$

$$2 \int \frac{\arctan x}{1+x^2} dx = \arctan^2 x$$

$$\int \frac{\arctan x}{1+x^2} dx = \frac{1}{2} \arctan^2 x + C.$$

7. $\int e^x . \sin x dx =$

$$u = \sin x \quad u' = \cos x, v' = e^x \quad v = e^x$$

$$= e^x . \sin x - \int e^x . \cos x dx$$

$=$

$$u = \cos x \quad u' = -\sin x, v' = e^x \quad v = e^x$$

$$= e^x . \sin x - \left(e^x . \cos x - \int (-\sin x) . e^x dx \right) = e^x . \sin x - e^x . \cos x - \int \sin x . e^x dx$$

Potom

$$\int e^x . \sin x dx = e^x . \sin x - e^x . \cos x - \int \sin x . e^x dx$$

$$2 \int e^x . \sin x dx = e^x . \sin x - e^x . \cos x$$

$$\int e^x . \sin x dx = \frac{e^x . \sin x - e^x . \cos x}{2} + c$$

Metoda substituční

8. $\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{1}{\sqrt{\ln x}} \cdot \frac{dx}{x} =$

$$\sqrt{\ln x} = t \Rightarrow \ln x = t^2 \Rightarrow \frac{dx}{x} = 2tdt$$

$$= 2\int \frac{1}{t} tdt = 2\int dt = 2t = 2\sqrt{\ln x} + C, \text{pro } x > 1.$$

9. $\int \frac{\sin x}{1+3\cos x} dx = \int \frac{1}{1+3\cos x} \sin x dx =$

$$1 + 3\cos x = t \Rightarrow -3\sin x dx = dt \Rightarrow \sin x dx = -\frac{1}{3}dt$$

$$= \int \frac{1}{t} \cdot \left(-\frac{1}{3}dt\right) = -\frac{1}{3}\int \frac{dt}{t} = -\frac{1}{3}\ln|t| = -\frac{1}{3}\ln|1+3\cos x| + C, \text{pro } x, \cos x \neq -\frac{1}{3}.$$

10. $\int \frac{\arctan x}{1+x^2} dx = \int \arctan x \cdot \frac{1}{1+x^2} dx =$

$$\arctan x = t \Rightarrow \frac{1}{1+x^2}dx = dt$$

$= \int tdt = \frac{t^2}{2} = \frac{\arctan^2 x}{2} + C.$

11. $\int \frac{\sin x \cos x}{\sqrt{2-\sin^2 x}} dx =$

$$\sqrt{2 - \sin^2 x} = t \Rightarrow 2 - \sin^2 x = t^2$$

$$-2\sin x \cos x dx = 2tdt \Rightarrow \sin x \cos x dx = -tdt$$

$= -\int \frac{tdt}{t} = -\int dt = -t = -\sqrt{2 - \sin^2 x} + C.$

12. $\int \frac{1}{x^2+2} dx =$

$$x = t\sqrt{2}, t = \frac{x}{\sqrt{2}} \Rightarrow dx = \sqrt{2}dt$$

$= \int \frac{\sqrt{2}dt}{2t^2+2} = \frac{\sqrt{2}}{2}\int \frac{1}{t^2+1}dt = \frac{\sqrt{2}}{2}\arctan \frac{x}{\sqrt{2}} + C.$

13. $\int \frac{1}{x^2+a^2} dx = \int \frac{1}{a^2(\frac{x^2}{a^2}+1)} dx = \frac{1}{a^2}\int \frac{1}{\frac{x^2}{a^2}+1} dx$

$$\frac{x}{a} = t, x = at \Rightarrow dx = adt$$

$= \frac{1}{a^2}\int \frac{adt}{t^2+1} = \frac{1}{a}\int \frac{1}{t^2+1}dt = \frac{1}{a}\arctan t = \frac{1}{a}\arctan \frac{x}{a} + C.$

14. $\int \frac{1}{x^2-2x+2}dx = \int \frac{1}{(x-1)^2+1}dx =$

$$x-1=t$$

$$dx=dt$$

$\int \frac{1}{t^2+1}dt = \arctan t = \arctan(x-1)+C.$

15. $\int \ln(x+\sqrt{1+x^2})dx =$

$$u'=1,\ v=\ln(x+\sqrt{1+x^2}),$$

$$u=x, v'=\frac{1+\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}.$$

$= x\ln(x+\sqrt{1+x^2}) - \int \frac{1}{\sqrt{1+x^2}}\cdot xdx =$

$$1+x^2=t \Rightarrow 2xdx=dt \Rightarrow xdx=\frac{1}{2}dt$$

$$= x\ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int t^{-\frac{1}{2}}dt = x\ln(x+\sqrt{1+x^2}) - t^{\frac{1}{2}} =$$

$$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2}+C.$$

16. $\int \frac{e^{2x}}{\sqrt[4]{e^x+1}}dx =$

$$\sqrt[4]{e^x+1}=t \Rightarrow e^x=t^4-1, x=\ln(t^4-1), dx=\frac{4t^3}{t^4-1}dt$$

$$=\int \frac{(t^4-1)^2}{t}\cdot\frac{4t^3}{t^4-1}dt=\int (t^4-1).4t^2dt = 4\int t^6-t^2dt = 4\left(\frac{t^7}{7}-\frac{t^3}{3}\right)+c = 4\left(\frac{1}{7}(e^x+1)^{\frac{7}{4}}-\frac{1}{3}(e^x+1)^{\frac{3}{4}}\right)+c$$

Integrace racionálních lomených funkcí

17. $\int \frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)}dx =$

$$\frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)} = \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{x-5} \Rightarrow A=0, B=0, C=-3, D=1$$

$$-3\int \frac{1}{(x-2)^3}dx+\int \frac{1}{x-5}dx = \frac{3}{2}\frac{1}{(x-2)^2}+\ln|x-5|+C, \text{pro } x \neq 2, x \neq 5.$$

18. $\int \frac{2x^3+5x^2+8}{2x^2+7x-15}dx =$

$$(2x^3+5x^2+8) : (2x^2+7x-15) = x-1+\frac{22x-7}{2x^2+7x-15}$$

$$= \int \left[x-1+\tfrac{22x-7}{2x^2+7x-15}\right]dx = \int \left[x-1+\tfrac{22x-7}{(2x-3)(x+5)}\right]dx$$

$$22x-7 = A(x+5)+B(2x-3) \Rightarrow A=4, B=9$$

$$= \int x dx - \int dx + 4\int \frac{1}{2x-3}dx + 9\int \frac{1}{x+5}dx = \frac{x^2}{2} - x + 2\ln|2x-3| + 9\ln|x+5| + C, \text{pro } x \neq \frac{3}{2}, x \neq -5.$$

19. $\int \frac{1}{x^3+x^2+2x+2}dx = \int \frac{1}{(x+1)(x^2+2)}dx$

$$\frac{1}{(x+1)(x^2+2)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2} \Rightarrow A=\frac{1}{3}, B=-\frac{1}{3}, C=\frac{1}{3}$$

$$= \frac{1}{3}\int \frac{1}{x+1} - \frac{1}{3}\int \frac{x-1}{x^2+2} = \frac{1}{3}\int \frac{1}{x+1} - \frac{1}{3}\int \frac{x}{x^2+2} + \frac{1}{3}\int \frac{1}{x^2+2} = \frac{1}{3}\ln|x+1| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x}{x^2+2} + \frac{\sqrt{2}}{6}\arctan\frac{x}{\sqrt{2}} =$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2+2) + \frac{\sqrt{2}}{6}\arctan\frac{x}{\sqrt{2}} + C, \text{pro } x \neq -1.$$

20. $\int \frac{5x+2}{x^2+2x+10}dx = \int \frac{5x+2}{(x+1)^2+9}dx =$

$$x+1 = 3t \Rightarrow dx = 3dt$$

$$= \int \frac{15t-3}{9t^2+9}3dt = \frac{3.3}{9}\int \frac{5t-1}{t^2+1}dt = \frac{5}{2}\int \frac{2t}{t^2+1}dt - \int \frac{1}{t^2+1}dt =$$

$$= \frac{5}{2}\ln(t^2+1) - \arctan t = \frac{5}{2}\ln\frac{x^2+2x+10}{9} - \arctan\frac{x+1}{3} + C.$$

21. $\int \frac{x+2}{x^3-2x^2+2x}dx = \int \frac{x+2}{x(x^2-2x+2)}dx =$

$$\frac{x+2}{x(x^2-2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2-2x+2} \Rightarrow A=1, B=-1, C=3.$$

$$= \int \left(\frac{1}{x} + \frac{-x+3}{x^2-2x+2}\right)dx = \int \frac{1}{x}dx - \int \frac{x-3}{x^2-2x+2}dx = \int \frac{1}{x}dx - \frac{1}{2}\int \frac{2x-2}{x^2-2x+2}dx - 2\int \frac{1}{x^2-2x+2}dx =$$

$$= \ln|x| - \ln(x^2-2x+2) - 2\int \frac{1}{x^2-2x+2}dx =$$

$$\int \frac{1}{x^2-2x+2}dx = \frac{1}{\sqrt{2-1^2}}\arctan\frac{x-1}{\sqrt{2-1^2}} = \arctan(x-1),$$

$$= \ln|x| + \ln(x^2-2x+2) - 2\arctan(x-1) + C = \ln\frac{|x|}{(x^2-2x+2)} - 2\arctan(x-1) + C, \text{pro } x \neq 0$$

Integrace iracionálních funkcí

22. $\int \frac{dx}{\sqrt{4x^2-3}} =$

$$\sqrt{4x^2-3} = t - 2x$$
$$4x^2 - 3 = t^2 - 4tx + 4x^2$$
$$-3 = t^2 - 4tx$$
$$x = \frac{t^2+3}{4t}dt$$
$$dx = \frac{t^2-3}{4t^2}dt$$

$$= \int \frac{\frac{t^2-3}{4t^2}}{t - 2\frac{t^2+3}{4t}}dt = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|2x+\sqrt{4x^2-3}|+C, \text{pro} |x| > \sqrt{\frac{3}{2}}.$$

23. $\int \frac{xdx}{\sqrt[3]{1+x}-\sqrt{1+x}} =$

$$1 + x = t^6 \Rightarrow dx = 6t^5dt, x = t^6 - 1$$

$$= \int \frac{t^6-1}{t^2-t^3}6t^5dt = -6\int \frac{t^6-1}{t-1}t^3 = -6\int t^8+t^7+t^6+t^5+t^4+t^3dt = -\frac{2}{3}t^9-\frac{3}{4}t^8-\frac{6}{7}t^7-t^6-\frac{6}{5}t^5-\frac{3}{2}t^4+c =$$
$$= -\tfrac{2}{3}(1+x)\sqrt{1+x}-\tfrac{3}{4}(1+x)\sqrt[3]{1+x}-\tfrac{6}{7}(1+x)\sqrt[6]{1+x}-1-x-\tfrac{6}{5}(1+x)^{\frac{5}{6}}-\tfrac{3}{2}(1+x)^{\frac{2}{3}}+c$$

24. $\int \frac{dx}{x+\sqrt{x^2+x+1}} =$

$$\sqrt{x^2+x+1} = t+x \Rightarrow x^2+x+1 = x^2+2tx+t^2, x = \frac{t^2-1}{1-2t}, dx = \frac{-2t^2+2t-2}{(1-2t)^2}dt$$

$$= \int \frac{\frac{-2t^2+2t-2}{(1-2t)^2}dt}{\frac{t^2-1}{1-2t}+t+\frac{t^2-1}{1-2t}} = \int \frac{2t^2-2t+2}{(t-2)(2t-1)} = \int 1+\frac{2}{t-2}-\frac{1}{2t-1}dt = t+2\ln|t-2|-\frac{1}{2}\ln|2t-1| =$$
$$= \sqrt{x^2+x+1}-x+2\ln|\sqrt{x^2+x+1}-x-2|-\frac{1}{2}\ln|2\sqrt{x^2+x+1-2x-1}|+c.$$

25. $\int \frac{dx}{x+\sqrt{x^2-x+1}} =$

$$\sqrt{x^2+x+1} = tx+1 \Rightarrow x^2+x+1 = t^2x^2+2tx+1, x = \frac{1+2t}{1-t^2}, dx = \frac{2t^2+2t+2}{(1-t^2)^2}dt$$

$$= \int \frac{\frac{2t^2+2t+2}{(1-t^2)^2}dt}{\frac{1+2t}{1-t^2}+t\frac{1+2t}{1-t^2}+1} = \int \frac{-2t^2-2t-2}{(t-1)(t+2)(t+1)^2}dt = \int \frac{-0.5}{t-1}-\frac{1.5}{t+1}+\frac{1}{(t+1)^2}+\frac{2}{t+2}dt =$$
$$= -0.5\ln|t-1| - 1.5\ln|t+1| - \frac{1}{t+1} + 2\ln|t+2| =$$
$$= -0.5\ln|\frac{\sqrt{x^2-x+1}-x-1}{x}|-1.5\ln|\frac{\sqrt{x^2-x+1}+x-1}{x}|-\frac{x}{\sqrt{x^2-x+1}+x-1}+2\ln|\frac{\sqrt{x^2-x+1}+2x-1}{x}|+c$$

26. $\int \frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} dx =$

$$\sqrt{x^2-1} = t + x \Rightarrow x^2 - 1 = t^2 + 2tx + x^2; x = \frac{-1-t^2}{2t}, dx = \frac{1-t^2}{2t^2} dt$$

$$= \int \frac{\frac{-1-t^2}{2t} + t + \frac{-1-t^2}{2t}}{\frac{-1-t^2}{2t} - \left(t + \frac{-1-t^2}{2t}\right)} \cdot \frac{1-t^2}{2t^2} dt = \int \frac{1-t^2}{2t^4} dt = \frac{1}{2} \int \frac{1}{t^4} - \frac{1}{t^2} dt = \frac{1}{2}\left(\frac{t^{-3}}{-3} - \frac{t^{-1}}{-1}\right) + c =$$

$$= -\frac{1}{6}\left(\sqrt{x^2-1} - x\right)^{-3} + \frac{1}{2}\left(\sqrt{x^2-1} - x\right)^{-1} + c$$

27. $\int \frac{\sqrt[3]{x}}{x+\sqrt{x}} dx =$

$$x = t^6 \Rightarrow dx = 6t^5 dt \Rightarrow t = \sqrt[6]{x}.$$

$$I = \int \frac{t^2}{t^6 + t^3} 6t^5 dt = 6 \int \frac{t^3 \cdot t^4}{t^3(t^3+1)} dt = 6 \int \frac{t^4}{t^3+1} dt =$$

$$t^4 : (t^3+1) = t - \frac{t}{t^3+1}$$

$$= 6 \int t - \frac{t}{t^3+1} dt = 6 \cdot \frac{t^2}{2} - 6 \int \frac{t}{t^3+1} dt = 3 \cdot t^2 - 6 \int \frac{t}{t^3+1} dt$$

Potřebujeme určit $\int \frac{t}{t^3+1} dt$.

$$\frac{t}{t^3+1} = \frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1} = \frac{A(t^2-t+1) + (Bt+C)(t+1)}{(t+1)(t^2-t+1)},$$

potom

$$t = A(t^2-t+1) + (Bt+C)(t+1),$$

po dosazení za t postupně $-1, 0, 1$ dostaneme

$$A = -\frac{1}{3}, B = C = \frac{1}{3}.$$

Potom

$$\int \frac{t}{t^3+1} dt = \int \frac{-\frac{1}{3}}{t+1} + \frac{\frac{1}{3}t + \frac{1}{3}}{t^2-t+1} dt = -\frac{1}{3}\left(\int \frac{1}{t+1} dt - \int \frac{t+1}{t^2-t+1} dt\right).$$

Určíme "modrý" a "červený" integrál a bude téměř hotovo.
Potřebujeme určit $\int \frac{1}{t+1} dt$. Toto je jednoduché:

$$\int \frac{1}{t+1} dt = \ln|t+1| + c.$$

A ještě $\int \frac{t+1}{t^2-t+1}dt$. Toto nebude úplně jednoduché. V čitateli se snažíme dostat derivaci jmenovatele:

$$\int \frac{t+1}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t+2}{t^2-t+1}dt = \frac{1}{2}\int \frac{2t-1+3}{t^2-t+1}dt,$$

funkci rozdělíme na dva zlomky, dostaneme dva integrály

$$\frac{1}{2}\left(\int \frac{2t-1}{t^2-t+1}dt + \int \frac{3}{t^2-t+1}dt\right).$$

Určíme $\int \frac{2t-1}{t^2-t+1}dt$ a $\int \frac{3}{t^2-t+1}dt$ a pak se vrátíme postupně až k "zelenému"integrálu.

- $\int \frac{2t-1}{t^2-t+1}dt = \ln(t^2-t+1)+c$. *Proč není nutná abs. hodnota argumentu?*
- $\int \frac{3}{t^2-t+1}dt = 3\int \frac{dt}{t^2-t+1} = 3\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}} = 3\int \frac{dt}{\frac{3}{4}\cdot\left(\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right)} = 4\int \frac{dt}{\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1} =$

$$\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = z \Rightarrow \frac{2}{\sqrt{3}}dt = dz \Rightarrow dt = \frac{\sqrt{3}}{2}dz.$$

Potom

$$= 4\int \frac{\frac{\sqrt{3}}{2}dz}{z^2+1} = 2\cdot\sqrt{3}\int \frac{dz}{z^2+1} = 2\cdot\sqrt{3}\arctan z + c = 2\cdot\sqrt{3}\arctan\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c = 2\cdot\sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)+c$$

Můžeme se vrátit k "červenému"integrálu:

$$\int \frac{t+1}{t^2-t+1}dt = \frac{1}{2}\left(\int \frac{2t-1}{t^2-t+1}dt + \int \frac{3}{t^2-t+1}dt\right) = \frac{1}{2}\left(\ln(t^2-t+1) + 2\cdot\sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right) =$$

$$= \frac{1}{2}\ln(t^2-t+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right).$$

A můžeme se vrátit k "zelenému"integrálu:

$$\int \frac{t}{t^3+1}dt = -\frac{1}{3}\left(\int \frac{1}{t+1}dt - \int \frac{t+1}{t^2-t+1}dt\right) = -\frac{1}{3}\left(\ln|t+1| - \frac{1}{2}\ln(t^2-t+1) - \sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right)+c$$

Potom

$$I = 3\cdot t^2 - 6\int \frac{t}{t^3+1}dt = 3\cdot t^2 + 2\cdot\left(\ln|t+1| - \frac{1}{2}\ln(t^2-t+1) - \sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(t-\frac{1}{2}\right)\right)+c.$$

Zůstává vrátit substituci $t = \sqrt[6]{x}$.

$$I = 3\cdot\sqrt[3]{x} + 2\cdot\left(\ln|\sqrt[6]{x}+1| - \frac{1}{2}\ln(\sqrt[3]{x} - \sqrt[6]{x}+1) - \sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(\sqrt[6]{x}-\frac{1}{2}\right)\right) + c.$$

Integrace goniometrických funkcí

28. $\int \frac{\sin^3 x dx}{\cos^2 x+1} =$

$$\cos x = t \Rightarrow -\sin x dx = dt$$

$$= \int \frac{(1-\cos^2 x).\sin x dx}{\cos^2 x+1} = -\int \frac{1-t^2}{1+t^2} dt = \int 1 - \frac{2}{t^2+1} dt = t - 2\arctan t =$$

$$= \cos x - 2\arctan(\cos x) + c.$$

29. $\int \frac{dx}{(2+\cos x)\sin x} =$

$$\tan\frac{x}{2} = t \Rightarrow \sin x = \frac{2t}{t^2+1}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2}dt$$

$$= \int \frac{\frac{2}{1+t^2}dt}{\left(2+\frac{1-t^2}{1+t^2}\right)\frac{2t}{1+t^2}} = \int \frac{1+t^2}{3+t^2} tdt = \frac{1}{3}\int \frac{2t}{3+t^2} + \frac{1}{t} dt = \frac{1}{3}\ln|(3+t^2)t| = \frac{1}{3}\ln|(3+\tan^2\frac{x}{2})\tan\frac{x}{2}| + c.$$

30. Integrál $\int \frac{dx}{\sin x}$ určíme čtyřmi různými způsoby:

- *1. způsob:*

$$\int \frac{dx}{\sin x} = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \frac{1}{2}\int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} dx =$$

$$\frac{x}{2} = t$$

$$dx = 2dt$$

$$= \int \frac{1}{\sin t\cos t} dt = \int \frac{\frac{1}{\cos^2 t}}{\frac{\sin t\cos t}{\cos^2 t}} dt = \int \frac{\frac{1}{\cos^2 t}}{\tan t} dt = \ln|\tan t| = \ln\left|\tan\frac{x}{2}\right| + C, \text{pro } x \neq k\pi, k \in Z.$$

- *2. způsob:*

$$\int \frac{dx}{\sin x} = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} dx + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} dx = -\ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right| + C, \text{pro } x \neq k\pi, k \in Z.$$

- *3. způsob:*

$$\int \frac{dx}{\sin x} = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{dx}{2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos\frac{x}{2}\cos\frac{x}{2}} = \int \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} dx =$$

$$\tan\frac{x}{2} = t$$

$$\frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} dx = dt$$

$$\int \frac{1}{t} dt = \ln|t| = \ln\left|\tan\frac{x}{2}\right| + C, \text{pro } x \neq k\pi, k \in Z.$$

- *4. způsob, univerzální substituce:*
  $\int \frac{dx}{\sin x} =$

$$\tan\frac{x}{2} = t, x = 2\arctan t$$

$$dx = 2\frac{1}{1+t^2}dt = \frac{2dt}{1+t^2}$$

$$\sin\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}, \cos\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\sin x = \sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos 2\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{1}{\frac{2t}{t^2+1}} \cdot \frac{2dt}{t^2+1} = \int \frac{dt}{t} = \ln|t| = \ln\left|\tan\frac{x}{2}\right| + C, \text{pro } x \neq k\pi, k \in Z.$$