



Section:

The Quest for de-Groot-like Dual of Pretopological Systems With Mathematica as of a Tool of Visualization

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Abstract. In this paper we study the preframe structure, representing the opens of a possible pretopological system, and its behavior with respect to the certain de Groot-like dualization construction. We present some counterexamples, contributing to the discussion regarding the possibility of obtaining similar results as there are known for the topological spaces. We also present a Mathematica 7 compatible package, which demonstrates and visualize the problem for the finite posets of opens.

Key words and phrases. Preframe, pretopological system, de Groot-like dual for preframes and pretopological systems.

Mathematics Subject Classification. Primary 6A12. Secondary 54B99, 54D30, 54E55.

1 Introduction and Status of the Problem

For a given topological space (X, τ) , a topology τ^d , generated by the family of all compact saturated sets used as its closed base, is called the de Groot dual of the original topology. Jimmie Lawson and Michael Mislove stated in [6] a problem, whether the sequence $\tau^d, \tau^{dd}, \tau^{ddd}, \dots$, containing the iterated duals of τ , is infinite or the process of taking duals terminates after finitely many steps with topologies that are dual to each other. The problem was solved by the second author, who in 2001 proved that for any topology it holds $\tau^{dd} = \tau^{ddd}$ [4] (the result was first announced and communicated on Toposym, in Prague 2001) and in 2004 the result was improved (again by the second author) to its final form $\tau^d = (\tau \vee \tau^{dd})^d$ [5]. Note that from the last mentioned equality it follows that for any topology, $\tau^d \subseteq \tau^{ddd}$. It should be also noted that the paper [4] pointed out several natural questions regarding the dual topologies. Some of them were addressed by the recent paper [8] of Tomoo Yokoyama. However, the second part of the original question of J. Lawson and M. Mislove stated in [6], which topologies can arise as duals, still remains open.

The questions of J. Lawson and M. Mislove related to the de Groot dual arise from an alternate approach to the certain constructions of various semantic models in the theoretical computer science, where the dual and the patch topologies constitute an important tool of investigation. However, there is a much wider class of related algebraic and topological structures, like topological systems, frames, locales, and many other, having even a greater importance for the topic than the class of topological spaces itself. An interesting direction of research was introduced by Bernhard Banaschewski [1], who replaced the usual frame structure (for example, of open sets in a topological space) by a more general, partially ordered structure called *preframe*, where the suprema exist for all non-empty up-directed subcollections. B. Banaschewski founded this structure useful for his alternate proof [1] of a relatively familiar result of P. Johnstone [3] - the localic version of the well-known Tichonov Theorem.

It is natural and potentially useful for applications in the theoretical computer science to study the preframe structure in connections with a proper modification of the de Groot dual. Although we do not have an analogue of the results of the second author reached for the topological spaces yet, some first attempts are already contained in the our paper [2], where we defined a counterpart of the de Groot dual for a certain class of pretopological systems. Pretopological systems form a slight generalization of the familiar notion of the topological systems (see, for example, [7] for the exact definition), where the frame structure of opens is replaced by the preframe structure. The pretopological systems, for which our modification of the de Groot dual is possible, are similar to the localic topological systems; also for them the abstract points of the system are fully determined by the structure of opens. We call these pretopological systems them *compactly-localic*. For the definitions and more detail, the reader is referred to [2].

In this paper we will concentrate on the preframe structure of the opens of the pretopological counterpart of the de Groot dual. It has been shown in [2], that under some circumstances, the opens of the dual may be represented as certain maps from A to the Sierpiński frame $\mathbf{2}$, where A is the poset representing the opens of the original pretopological system. If we denote the poset of such maps by $\langle A \rightarrow \mathbf{2} \rangle$, the sequence of the iterated duals then have the form of $\langle A \rightarrow \mathbf{2} \rangle$, $\langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$, $\langle \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$, \dots , etc. So far we do not know whether there exist (and hold) appropriate, full-featured counterparts of the results of the second author for the pretopological systems and how they should look. But some preliminary results and counterexamples presented in this paper illustrate the difficulties which should be overwhelmed in order to reach some final, positive result.

In parallel to the theoretical part of the paper, we present here also a Mathematica 7 compatible package `IteratedDuals.m` which calculates the first three elements of the previously mentioned sequence representing the iterated duals for a finite poset A . The package also displays their Hasse diagrams. Although the package itself has no theoretical importance for our investigation, it can serve as a useful visualization and demonstration tool, conducing to the reader's convenience and comfort. Another, alternate reason for presenting of the package (at the conference Aplimat) is to move the usage of the software Mathematica somewhat towards to the more theoretical disciplines, and demonstrate its utility - even for such theoretical disciplines, as topology.

Now, let us recall some notions and make some denotations. We say that a partially ordered set (briefly a poset) A is a *preframe* [1], if A is closed under directed joins and finite meets (including the meet of the empty set), such that the binary meets distribute over the directed joins. It should be noted that by the usual definition, a directed set is non-empty, so the

preframe need not have the least element - the supremum of the empty set (that is, $\bigvee \emptyset$). On the other hand, a preframe always has the greatest element $\bigwedge \emptyset$. By $\mathbf{2} = \{\perp, \top\}$ we denote the *Sierpiński frame*, consisting of the two elements, \top and \perp . Let A be a set, then each mapping $f : A \rightarrow \mathbf{2}$ can be uniquely identified with its \top -kernel, $\text{Ker}_\top f = \{x \mid x \in A, f(x) = \top\}$. In this way, we can equip $\mathbf{2}^A$ with the partial order, given by the inclusion on the power set 2^A . By **False** and **True** we denote the constant functions on A identically equal to \perp and \top , respectively. Let A, B be posets. We say that a mapping $f : A \rightarrow B$ is a *morphism* if it preserves the (non-empty) directed joins and finite meets (including the meet of the empty set), whenever they exist. The set of all morphisms $f : A \rightarrow \mathbf{2}$ we denote by $\langle A \rightarrow \mathbf{2} \rangle$. We consider it as a poset, naturally equipped with the order induced from the partially ordered set $\mathbf{2}^A$. Throughout this paper, if not otherwise stated, by the dual of a poset A we mean the set $\langle A \rightarrow \mathbf{2} \rangle$ with this order.

2 Theoretical Results

We will start with the following proposition.

Proposition 2.1 *Let A be a poset. Then $\langle A \rightarrow \mathbf{2} \rangle$ forms a preframe of all morphisms of A to $\mathbf{2}$.*

Proof. We will show that $\langle A \rightarrow \mathbf{2} \rangle$ has the non-empty directed joins, all finite meets (including the meet of the empty set) and that the meets distribute over the directed joins.

Let $Y \subseteq \langle A \rightarrow \mathbf{2} \rangle$ be non-empty and directed. Let $f(a) = \bigvee_{y \in Y} y(a)$ for every $a \in A$. We will show that $f = \bigvee Y$ in $\langle A \rightarrow \mathbf{2} \rangle$. First, we must show that $f \in \langle A \rightarrow \mathbf{2} \rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A . Then $f(\bigvee B) = \bigvee_{y \in Y} y(\bigvee B) = \bigvee_{y \in Y} \bigvee_{b \in B} y(b) = \bigvee_{b \in B} \bigvee_{y \in Y} y(b) = \bigvee_{b \in B} f(b)$, so f preserves non-empty directed joins. Let $C \subseteq A$ be non-empty and finite. Suppose that $\bigwedge C$ exists in A . Then $f(\bigwedge C) = \bigvee_{y \in Y} y(\bigwedge C) = \bigvee_{y \in Y} \bigwedge_{c \in C} y(c) = \top$ implies that there exist some $y_1 \in Y$, such that for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigwedge_{c \in C} \bigvee_{y \in Y} y(c) = \bigwedge_{c \in C} f(c)$ which implies $f(\bigwedge C) \leq \bigwedge_{c \in C} f(c)$. Conversely, suppose that $\bigwedge_{c \in C} f(c) = \bigwedge_{c \in C} \bigvee_{y \in Y} y(c) = \top$. Then for every $c \in C$ there is some $y_c \in Y$ with $y_c(c) = \top$. Since Y is directed and C is finite, there exist some $y_1 \in Y$ such that $y_1 \geq y_c$ for every $c \in C$. Hence, for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigvee_{y \in Y} \bigwedge_{c \in C} y(c) = \bigvee_{y \in Y} y(\bigwedge C) = f(\bigwedge C)$ which implies that $f(\bigwedge C) \geq \bigwedge_{c \in C} f(c)$. Now we have $f(\bigwedge C) = \bigwedge_{c \in C} f(c)$, so f preserves also non-empty finite meets. It remains to check the preservation of the empty meet. Suppose that A has the greatest element $\bigwedge \emptyset \in A$. Then $f(\bigwedge \emptyset) = \bigvee_{y \in Y} y(\bigwedge \emptyset) = \bigvee_{y \in Y} \top = \top$. Hence, f is an element of $\langle A \rightarrow \mathbf{2} \rangle$, and, clearly, an upper bound of Y in $\langle A \rightarrow \mathbf{2} \rangle$. Now, let $u \in \langle A \rightarrow \mathbf{2} \rangle$ be another upper bound of Y . Then, for every $a \in A$ and every $y \in Y$ it follows that $u(a) \geq y(a)$, which gives $u(a) \geq \bigvee_{y \in Y} y(a) = f(a)$ and, consequently, $u \geq f$. So f is a correctly defined supremum of Y in $\langle A \rightarrow \mathbf{2} \rangle$.

Suppose that $Z \subseteq \langle A \rightarrow \mathbf{2} \rangle$ is non-empty and finite. Let $g(a) = \bigwedge_{z \in Z} z(a)$ for every $a \in A$. We will show that $g = \bigwedge Z$ in $\langle A \rightarrow \mathbf{2} \rangle$. First, we must show that $g \in \langle A \rightarrow \mathbf{2} \rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A . Then $g(\bigvee B) = \bigwedge_{z \in Z} z(\bigvee B) = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \top$ implies that for every $z \in Z$ there is $b_z \in B$ with $z(b_z) = \top$. Since Z is finite and B is directed, there is some $b_1 \in B$ such that $b_1 \geq b_z$ for every $z \in Z$. Then $z(b_1) = \top$ for every $z \in Z$, which implies that $\top = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \bigvee_{b \in B} g(b)$. Hence, $g(\bigvee B) \leq \bigvee_{b \in B} g(b)$.

Conversely, suppose that $\bigvee_{b \in B} g(b) = \bigvee_{b \in B} \bigwedge_{z \in Z} z(b) = \top$. Then, there exists $b_1 \in B$, such that $z(b_1) = \top$ for every $z \in Z$. Then $\top = \bigwedge_{z \in Z} \bigvee_{b \in B} z(b) = \bigwedge_{z \in Z} z(\bigvee B) = g(\bigvee B)$. It follows that $g(\bigvee B) \geq \bigvee_{b \in B} g(b)$ and hence, together with the previously proved (converse) inequality, we have $g(\bigvee B) = \bigvee_{b \in B} g(b)$. Now, let $C \subseteq A$ be non-empty and finite, having $\bigwedge C \in A$. Then $g(\bigwedge C) = \bigwedge_{z \in Z} z(\bigwedge C) = \bigwedge_{z \in Z} \bigwedge_{c \in C} z(c) = \bigwedge_{c \in C} \bigwedge_{z \in Z} z(c) = \bigwedge_{c \in C} g(c)$. Finally, suppose that A has the greatest element $\bigwedge \emptyset \in A$. Then $g(\bigwedge \emptyset) = \bigwedge_{z \in Z} z(\bigwedge \emptyset) = \bigwedge_{z \in Z} \top = \top$. It follows that g is an element of $\langle A \rightarrow \mathbf{2} \rangle$, and, clearly, a lower bound of Z in $\langle A \rightarrow \mathbf{2} \rangle$. Let $l \in \langle A \rightarrow \mathbf{2} \rangle$ be a lower bound of Z . Then, for every $a \in A$ and every $z \in Z$ we have $l(a) \leq z(a)$, which gives $l(a) \leq \bigwedge_{z \in Z} z(a) = g(a)$ and, consequently, $l \leq g$. Therefore, g is a correctly defined infimum of Z in $\langle A \rightarrow \mathbf{2} \rangle$. Moreover, the mapping **True**, constantly equal to \top , obviously preserves all non-empty directed joins and all finite meets, so $\langle A \rightarrow \mathbf{2} \rangle$ also has the greatest element. Note that **True** does not preserve the empty join, but it is not required.

Finally, we will show that binary meets distribute over directed joins in $\langle A \rightarrow \mathbf{2} \rangle$. Let $x \in \langle A \rightarrow \mathbf{2} \rangle$ and $Y \subseteq \langle A \rightarrow \mathbf{2} \rangle$ be directed. Then $(x \wedge (\bigvee Y))(a) = x(a) \wedge (\bigvee Y)(a) = x(a) \wedge (\bigvee_{y \in Y} y(a)) = \bigvee_{y \in Y} (x(a) \wedge y(a)) = \bigvee_{y \in Y} ((x \wedge y)(a)) = (\bigvee_{y \in Y} (x \wedge y))(a)$ for every $a \in A$, which implies $x \wedge (\bigvee Y) = \bigvee_{y \in Y} (x \wedge y)$. By the definition, $\langle A \rightarrow \mathbf{2} \rangle$ is a preframe.

Note that it may happen that $\langle A \rightarrow \mathbf{2} \rangle$ has the bottom $\perp = \bigvee \emptyset$ although the constant mapping **False** : $A \rightarrow \mathbf{2}$ with the constant value \perp need not be an element of $\langle A \rightarrow \mathbf{2} \rangle$. Let A be a poset. Let us denote by $h_A : A \rightarrow \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$ a mapping for which $h_A(a)(x) = x(a)$ for every $x \in \langle A \rightarrow \mathbf{2} \rangle$.

Proposition 2.2 *Let A be a poset. Then $h_A : A \rightarrow \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$ is a morphism.*

Proof. Suppose that there exists the greatest element $\bigwedge \emptyset \in A$. It follows that $h_A(\bigwedge \emptyset)(x) = x(\bigwedge \emptyset) = \top$ for every morphism $x \in \langle A \rightarrow \mathbf{2} \rangle$, so $h_A(\bigwedge \emptyset) = \top$.

Let $B \subseteq A$ be non-empty and directed and suppose that there exists $\bigvee B \in A$. Let $x \in \langle A \rightarrow \mathbf{2} \rangle$. Then $h_A(\bigvee B)(x) = x(\bigvee B) = \bigvee_{b \in B} x(b) = \bigvee_{b \in B} h_A(b)(x) = (\bigvee_{b \in B} h_A(b))(x)$, which implies that $h_A(\bigvee B) = \bigvee_{b \in B} h_A(b)$.

Let $C \subseteq A$ be non-empty, finite and assume that there exists $\bigwedge C \in A$. Let $x \in \langle A \rightarrow \mathbf{2} \rangle$. It follows $h_A(\bigwedge C)(x) = x(\bigwedge C) = \bigwedge_{c \in C} x(c) = \bigwedge_{c \in C} h_A(c)(x) = (\bigwedge_{c \in C} h_A(c))(x)$, which implies that $h_A(\bigwedge C) = \bigwedge_{c \in C} h_A(c)$.

Since h_A preserves all non-empty directed joins and all finite meets, it follows that h_A is a morphism.

Example 2.1 *There exist a preframe A such that h_A is not an epimorphism.*

Construction. Let $A = \omega + 1 = \{1, 2, \dots, \omega\}$, where ω is the first infinite ordinal, with its natural linear order. Let $n' : A \rightarrow \mathbf{2}$ be a mapping with the \top -kernel $\{n, n + 1, \dots, \omega\}$ for every $n \in \mathbb{A}$ and $(\omega + 1)'$ be a mapping identically equal to \perp . The construction is illustrated by the figure:

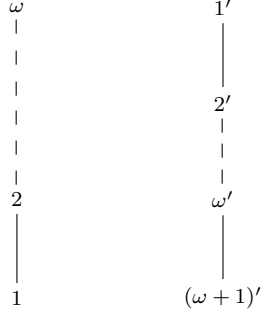


Figure 1.

Since every morphism is an isotone mapping and the constant mapping with the empty \top -kernel, $(\omega + 1)' = \mathbf{False}$, is not a morphism, it is not difficult to check that $\langle A \rightarrow \mathbf{2} \rangle = \{\omega', \dots, 2', 1'\}$. Notice that $\langle A \rightarrow \mathbf{2} \rangle$ is linearly ordered by the set inclusion of the corresponding \top -kernels of its elements. For every $x \in \langle A \rightarrow \mathbf{2} \rangle$ we put

$$p(x) = \begin{cases} \top, & \text{for } x > \omega' \\ \perp, & \text{for } x = \omega'. \end{cases}$$

Obviously p is a morphism, so $p \in \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$. But for every $a \in A$ and every $x \in \langle A \rightarrow \mathbf{2} \rangle$ it follows

$$h_A(a)(x) = x(a) = \begin{cases} \top & \text{for } x \geq a' \\ \perp & \text{for } x < a'. \end{cases}$$

Therefore, there is no $a \in A$ such that $p = h_A(a)$, which implies that h_A is not a surjection.

Example 2.2 *There exist a preframe A such that h_A is not a monomorphism.*

Construction. Let $A = \{0, 1, \dots, \omega\}$ be the distributive lattice with the Hasse diagram given by the figure:

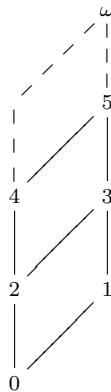


Figure 2.

That means, 0 is the bottom, ω is the top, $2k$ has two successors $2k + 1$, $2k + 2$ and $2k + 1$ has a unique successor $2k + 3$ for every $k \in \{0, 1, \dots\}$. Since for every $Y \subseteq A$ infinite it follows

$\bigvee Y = \omega$, the binary meets distribute over all joins. It follows that A is preframe (moreover, a frame).

Let $x \in \langle A \rightarrow \mathbf{2} \rangle$. We will show that $x(0) = x(1)$. If $x(0) = \top$, then also $x(1) = x(0 \vee 1) = x(0) \vee x(1) = \top \vee x(1) = \top$. Suppose that $x(2k) = \perp$ for every $k \in \{0, 1, \dots\}$. The set $S = \{2, 4, \dots\}$ infinite and directed. It follows that $x(\omega) = x(\bigvee S) = \bigvee_{s \in S} x(s) = \bigvee_{s \in S} \perp = \perp$. Then $x(1) = x(1 \wedge \omega) = x(1) \wedge x(\omega) = x(1) \wedge \perp = \perp$. Finally, suppose that k is the greatest number from $\{0, 1, \dots\}$ such that $x(2k) = \perp$. But $\perp = x(2k) = x((2k+2) \wedge (2k+1)) = x(2k+2) \wedge x(2k+1) = \top \wedge x(2k+1)$, which implies $x(2k+1) = \perp$. Then $x(1) = x(1 \wedge (2k+1)) = x(1) \wedge x(2k+1) = x(1) \wedge \perp = \perp$. Therefore, $h_A(0) = h_A(1)$ which implies that h_A is not injective.

Proposition 2.3 *Let A be a finite preframe. Then $h_A : A \rightarrow \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$ is an isomorphism.*

Proof. Let $p \in \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$. We put $x_1 = \bigwedge \text{Ker}_\top(p)$ and $a_1 = \bigwedge \text{Ker}_\top(x_1)$, where Ker_\top is a denotation for the \top -kernel of a mapping with its co-domain equal to a subset of $\mathbf{2}$. Since A is finite, also $\text{Ker}_\top(p)$ is a finite set; say $\text{Ker}_\top(p) = \{y_1, y_2, \dots, y_k\}$. Then $p(x_1) = p(y_1 \wedge y_2 \wedge \dots \wedge y_k) = p(y_1) \wedge p(y_2) \wedge \dots \wedge p(y_k) = \top$, which means that x_1 is the least element of $\text{Ker}_\top(p)$. Similarly, a_1 is the least element of $\text{Ker}_\top(x_1)$.

We claim that $h_A(a_1) = p$. Indeed, for every $y \in \langle A \rightarrow \mathbf{2} \rangle$ it follows $p(y) = \top \Leftrightarrow y \in \text{Ker}_\top(p) \Leftrightarrow x_1 \leq y \Leftrightarrow \text{Ker}_\top(x_1) \subseteq \text{Ker}_\top(y) \Leftrightarrow a_1 \in \text{Ker}_\top(y) \Leftrightarrow h_A(a_1)(y) = y(a_1) = \top$. Hence, $h_A(a_1) = p$ which means that h_A is surjective.

Suppose that there exist $a_2 \in A$ such that also $h_A(a_2) = p$. Then $x_1(a_2) = \top$, which implies that $a_1 \leq a_2$. Suppose that $a_1 \neq a_2$ and let

$$z(a) = \begin{cases} \top, & \text{for } a \geq a_2 \\ \perp, & \text{otherwise.} \end{cases}$$

Since A is finite, z obviously preserves all directed joins. Then $z(\bigwedge \emptyset) = \top$ since $a_1 < a_2 \leq \bigwedge \emptyset$. Let $a, b \in A$. Then $z(a \wedge b) = \top \Leftrightarrow a \wedge b \geq a_2 \Leftrightarrow a \geq a_2$ and $b \geq a_2 \Leftrightarrow z(a) = \top$ and $z(b) = \top \Leftrightarrow z(a) \wedge z(b) = \top$. Hence, $z(a \wedge b) = z(a) \wedge z(b)$. It follows that z preserves also all finite meets. Then $z \in \langle A \rightarrow \mathbf{2} \rangle$, but $h_A(a_1)(z) = z(a_1) = \perp \neq \top = z(a_2) = h_A(a_2)(z)$, which is a contradiction. Therefore, $a_1 = a_2$, which implies that h_A is injective.

The following corollary is an immediate consequence of the previous proposition and Proposition 2.1.

Corollary 2.3 *Let A be a finite poset. Then its iterated duals, $\langle A \rightarrow \mathbf{2} \rangle$ and $\langle \langle \langle A \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle \rightarrow \mathbf{2} \rangle$, are isomorphic.*

3 Interactions with Mathematica

For a visualization of the problem, we have developed a Mathematica package `IteratedDuals.m`, which can display the Hasse diagrams of the original, finite poset and its first three iterated duals. There are certainly several possibilities how to represent a poset in Mathematica. In our package, a poset is considered as embedded to a suitable power set, partially ordered by the

inclusion. This representation of posets is well compatible with the approach used in the theoretical part of this paper, because the set of all maps from a poset A to the Sierpiński frame $\mathbf{2}$ is naturally frame-isomorphic with the power set $\mathbf{2}^A$ equipped with the inclusion partial order.

In the package `IteratedDuals.m`, the elements of $\langle A \rightarrow \mathbf{2} \rangle$ are represented in two complementary formats. In the first one, the map $f : A \rightarrow \mathbf{2}$ is represented as a binary relation, that is, as an element of $A \times \mathbf{2}$. This representation is more suitable for verifying whether a map f is a morphism; and since all our considered sets are finite, only the preservation of finite meets (including the meet of the empty set) is necessary to check. In the package, the “is-morphism-test” is provided by a combination of the following functions:

```
(* Check if an element is a lower bound of a set *)
IsLowerBound2TF[Element_, ElementSet_List]:=Module[{n,i, Passed},
n=Length[ElementSet];
Passed=True;
For[i=1, i<=n,i++,If[Element ∩ ElementSet[[i]]!=Element,Passed=False]];
Return[Passed]];

(* Find all lower bounds of a set *)
LowerBounds[ElementSet_List, OrderedSet_List]:=Module[{n,i, LBounds},
n=Length[OrderedSet];
LBounds={};
For[i=1,i<=n,i++,If[IsLowerBound2TF[OrderedSet[[i]], ElementSet],
LBounds=Append[LBounds, OrderedSet[[i]] ]]];
Return[Union[LBounds]] ];

(* Check whether the infimum of a set in an ordered set exists *)
MeetExists[ElementSet_List,OrderedSet_List]:=Module[{LBounds,MeetCandidate,ESet, MExists},
MExists=False;
ESet=Union[ElementSet];
LBounds=LowerBounds[ESet, OrderedSet];
MeetCandidate=Union[Flatten[LBounds,1]];
If[ESet ∩ OrderedSet==ESet,If[MemberQ[LBounds,MeetCandidate],MExists=True]];
Return[MExists] ];

(* Calculate the infimum - meet of a set in an ordered set *)
Meet[ElementSet_List,OrderedSet_List]:=Module[{LBounds,MeetCandidate,ESet},
ESet=Union[ElementSet];
LBounds=LowerBounds[ESet, OrderedSet];
MeetCandidate=Union[Flatten[LBounds,1]];
If[ESet ∩ OrderedSet==ESet,If[MemberQ[LBounds, MeetCandidate],
Return[MeetCandidate]]]];

(* Check if a map is a morphism *)
IsMorphism2TF[Mapping_List]:=Module[{X, i, j, n, MeetTemp, Passed},
X=Domain[Mapping];
n=Length[Mapping];
Passed=True;
For[i=1, i<=n,i++,
For[j=1, j<=n, j++,
If[MeetExists[{Mapping[[i]][[1]],Mapping[[j]][[1]]}, X],
If[!MemberQ[Mapping,{Meet[{Mapping[[i]][[1]],Mapping[[j]][[1]]}, X],
Mapping[[i]][[2]] ^ Mapping[[j]][[2]] }], Passed=False] ]];
If[MeetExists[{}],X],
If[!MemberQ[Mapping, {Meet[{}],X},True]], Passed=False]]];
```

```
Return[Passed] ];
```

In the second format, the map $f : A \rightarrow \mathbf{2}$ is represented by its \top -kernel (which coincides, in this case, with another notion, in some literature referred as the *support* of f) – the set of all elements in A for which the value of f is \top . This representation is more suitable and efficient for working with the partial order on $\langle A \rightarrow \mathbf{2} \rangle$, and displaying the corresponding Hasse diagrams. For the conversion from the first format to the second one, we use the following functions:

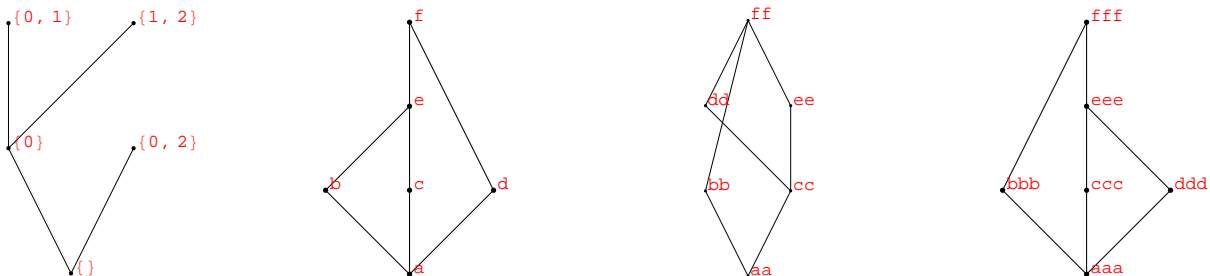
```
(* Find the support of a map *)
Support[Mapping_List]:=Module[{Function,n,i, Supp},
Function=Union[Mapping];
n=Length[Function];
Supp={};
For[i=1, i<=n,i++,If[Function[[i]][[2]]==True, Supp=Append[Supp,
Function[[i]][[1]]]];
Return[Union[Supp]] ];

(* Convert maps to their supports *)
Maps2Supports[x_List]:=Module[{Maps, n, SetofSupports},
Maps=Union[x];
n=Length[Maps];
SetofSupports={};
For[i=1, i<=n,i++, SetofSupports=Append[SetofSupports,Support[Maps[[i]]] ]];
Return[Union[SetofSupports]] ];
```

For displaying the Hasse diagrams, our package internally uses the functions `HasseDiagram` and `ShowLabeledGraph`, which are included in the standard Mathematica package `Combinatorica`. This package is automatically loaded with `IteratedDuals`. Below there is an example of the usage:

```
<< IteratedDuals`
```

```
Poset = {{0, 1}, {1, 2}, {2, 0}, {0}, {}};
ShowDuals[Poset]
Poset and its three iterated duals:
```



The Legend:

The elements of the first dual:

```
a={}
b={{0, 1}}
c={{0, 2}}
```



```

d={{1, 2}}
e={{0}, {0, 1}, {0, 2}}
f={{}, {0}, {0, 1}, {0, 2}, {1, 2}}

```

The elements of the second dual:

```

aa={{ {}, {0}, {0, 1}, {0, 2}, {1, 2}}}
bb={{ {1, 2}, {}, {0}, {0, 1}, {0, 2}, {1, 2}}}
cc={{ {0}, {0, 1}, {0, 2}, {}, {0}, {0, 1}, {0, 2}, {1, 2}}}
dd={{ {0, 1}, {0}, {0, 1}, {0, 2}, {}, {0}, {0, 1}, {0, 2}, {1, 2}}}
ee={{ {0, 2}, {0}, {0, 1}, {0, 2}, {}, {0}, {0, 1}, {0, 2}, {1, 2}}}
ff={{ {}, {0, 1}, {0, 2}, {1, 2}, {0}, {0, 1}, {0, 2}, {}, {0}, {0, 1}, {0, 2}, {1, 2}}}

```

...

(the output is shortened)

Note that the package `IteratedDuals` can be installed by the usual way to the appropriate directory and it is compatible with Mathematica 7.0. The package will be freely available and included with the Aplimat conference materials.

4 Conclusion

Although for a finite poset A we have an adequate counterpart of the result $\tau^d \subseteq \tau^{ddd}$ proved by the second author for the topological spaces (as it is shown in Proposition 2.3), the counterexamples in Example 2.1 and Example 2.2 demonstrate that a requested general result cannot be reached in this form. The reason for the negative conclusion possibly could lie in the fact, that unlike in the topological case, the points of the corresponding pretopological system are modified in each step of taking the dual. The fact of modifying the underlying set of points we should take into our considerations and appropriately adjust the way how the posets of dualized opens are compared. We will concentrate on this topic in more detail in our forthcoming paper.

Acknowledgement

This research is supported by the research intention of the Ministry of Education of the Czech Republic MSM0021630503 (MIKROSYN).

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