De Grootova dualizace v usměrněně kompletních topologických strukturách II

De Groot dualization in directly complete topological structures II

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1. Introduction

Let (X, τ) be a topological space. Recall that a topology τ^d, generated by the family of all compact saturated sets used as its closed base, is called the de Groot dual of the topology τ. In a paper (12) J. D. Lawson and M. Mislove stated a question whether a sequence of iterated de Groot duals of any topology terminates by two topologies, which are dual to each other. In his oral communication at the International Topology Conference in Istanbul in 2000, B. Burdick mentioned the Lawson's and Mislove's problem again and presented a positive, partial solution of the problem for some topologies on hyperspaces (2). In 2001, the second author proved that for any topology τ it holds $\tau^{dd} = \tau^{dddd}$ (9) (first announced and communicated on Toposym, Prague 2001). In 2004 the second author improved his result to its so far final form $\tau^d = (\tau \vee \tau^{dd})^d$ (11).

The topological formulation of the problem of J. D. Lawson and M. Mislove has its origins at certain constructions of various semantic models in the theoretical computer science, for which the dual and path topologies have some basic importance. However, the topological approach is not the only possible, and there are plenty of examples of mathematical and algebraic structures useful in computer science and domain theory, whose behavior and properties cannot be simply described by the classic topologies. Hence, it seems to be natural and interesting to study the original question of J. D. Lawson and M. Mislove also from a modified point of view. In this paper we will continue in our considerations regarding seeking the de Groot dual analogue for topological structures with opens having the *preframe* structure in the sense of B. Banaschewski, that we presented a year ago at the previous International Colloquium (3) and at the last International Conference Aplimat in Bratislava (4). Our paper (3) is concentrated especially on the construction of the appropriate analogue of the de Groot dual for a certain class of so called pretopological systems, a structure which is similar to topological systems (described, for example, in (13)) but the frame structure of opens is replaced by the (more general) preframe of opens. In (4) we study especially the dualization properties of the preframe of opens itself and give several counterexamples.

2. Main Results

At first, let us make some useful denotations and recall some notions. We say that a poset A is a *preframe* (introduced by B. Banaschewski in (1)), if A is closed under directed joins and finite meets (including the meet of the empty set), such that the binary meets distribute over the directed joins. Recalling the usual definition, every directed set is non-empty, so the preframe need not have the least element $\bigvee \emptyset$ - the supremum of the empty set. On the other hand, a preframe always has the greatest element $\bigwedge \emptyset$. By $\mathbf{2} = \{\bot, \top\}$ we denote the *Sierpiński frame*, consisting of the two elements, $\top$ and $\bot$. Let A be a set, then each mapping $f : A \to \mathbf{2}$

can be uniquely identified with its $\top$-kernel, $\mathrm{Ker}_\top f = \{x | x \in A, f(x) = \top\}$. In this way, we can equip $\mathbf{2}^A$ with the partial order, given by the inclusion on the power set 2^A. By **False** and **True** we denote the constant functions on A identically equal to $\bot$ and $\top$, respectively.

For our further considerations, we will need the following lemma:

Lemma 1. Let A be a poset. Then the following conditions hold:

(i) $\langle A \to \mathbf{2}\rangle$ forms a preframe of all morphisms of A to $\mathbf{2}$.

(ii) For a directed set $Y \subseteq \langle A \to \mathbf{2}\rangle$, and $a \in A$, it holds $(\bigvee Y)(a) = \bigvee_{y\in Y} y(a)$.

(iii) For a non-empty finite set $Z \subseteq \langle A \to \mathbf{2}\rangle$, and $a \in A$, $(\bigwedge Z)(a) = \bigwedge_{z\in Z} z(a)$.

(iv) $\bigwedge \emptyset$, the top element of $\langle A \to \mathbf{2}\rangle$, is represented by the constant mapping $\mathbf{True} : A \to \mathbf{2}$, identically equal to $\top$.

Proof. First, let us prove (ii). Let $Y \subseteq \langle A \to \mathbf{2}\rangle$ be non-empty and directed. Let $f(a) = \bigvee_{y\in Y} y(a)$ for every $a \in A$. We will show that $f = \bigvee Y$ in $\langle A \to \mathbf{2}\rangle$. First, we must prove that $f \in \langle A \to \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A. Then $f(\bigvee B) = \bigvee_{y\in Y} y(\bigvee B) = \bigvee_{y\in Y}\bigvee_{b\in B} y(b) = \bigvee_{b\in B}\bigvee_{y\in Y} y(b) = \bigvee_{b\in B} f(b)$, so f preserves non-empty directed joins. Let $C \subseteq A$ be non-empty and finite. Suppose that $\bigwedge C$ exists in A. Then $f(\bigwedge C) = \bigvee_{y\in Y} y(\bigwedge C) = \bigvee_{y\in Y}\bigwedge_{c\in C} y(c) = \top$ implies that there exist some $y_1 \in Y$, such that for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigwedge_{c\in C}\bigvee_{y\in Y} y(c) = \bigwedge_{c\in C} f(c)$ which implies $f(\bigwedge C) \leq \bigwedge_{c\in C} f(c)$. Conversely, suppose that $\bigwedge_{c\in C} f(c) = \bigwedge_{c\in C}\bigvee_{y\in Y} y(c) = \top$. Then for every $c \in C$ there is some $y_c \in Y$ with $y_c(c) = \top$. Since Y is directed and C is finite, there exist some $y_1 \in Y$ such that $y_1 \geq y_c$ for every $c \in C$. Hence, for every $c \in C$ it follows $y_1(c) = \top$. Then $\top = \bigvee_{y\in Y}\bigwedge_{c\in C} y(c) = \bigvee_{y\in Y} y(\bigwedge C) = f(\bigwedge C)$ which implies that $f(\bigwedge C) \geq \bigwedge_{c\in C} f(c)$. Now we have $f(\bigwedge C) = \bigwedge_{c\in C} f(c)$, so f preserves also non-empty finite meets. It remains to check the preservation of the empty meet. Suppose that A has the greatest element $\bigwedge\emptyset \in A$. Then $f(\bigwedge\emptyset) = \bigvee_{y\in Y} y(\bigwedge\emptyset) = \bigvee_{y\in Y}\top = \top$. Hence, f is an element of $\langle A \to \mathbf{2}\rangle$, and, clearly, an upper bound of Y in $\langle A \to \mathbf{2}\rangle$. Now, let $u \in \langle A \to \mathbf{2}\rangle$ be another upper bound of Y. Then, for every $a \in A$ and every $y \in Y$ it follows that $u(a) \geq y(a)$, which gives $u(a) \geq \bigvee_{y\in Y} y(a) = f(a)$ and, consequently, $u \geq f$. So f is a correctly defined supremum of Y in $\langle A \to \mathbf{2}\rangle$.

Now, let us show (iii). Suppose that $Z \subseteq \langle A \to \mathbf{2}\rangle$ is non-empty and finite. Let $g(a) = \bigwedge_{z\in Z} z(a)$ for every $a \in A$. We will show that $g = \bigwedge Z$ in $\langle A \to \mathbf{2}\rangle$. First, we must show that $g \in \langle A \to \mathbf{2}\rangle$. Let $B \subseteq A$ be non-empty and directed, such that $\bigvee B$ exists in A. Then $g(\bigvee B) = \bigwedge_{z\in Z} z(\bigvee B) = \bigwedge_{z\in Z}\bigvee_{b\in B} z(b) = \top$ implies that for every $z \in Z$ there is $b_z \in B$ with $z(b_z) = \top$. Since Z is finite and B is directed, there is some $b_1 \in B$ such that $b_1 \geq b_z$ for every $z \in Z$. Then $z(b_1) = \top$ for every $z \in Z$, which implies that $\top = \bigvee_{b\in B}\bigwedge_{z\in Z} z(b) = \bigvee_{b\in B} g(b)$. Hence, $g(\bigvee B) \leq \bigvee_{b\in B} g(b)$. Conversely, suppose that $\bigvee_{b\in B} g(b) = \bigvee_{b\in B}\bigwedge_{z\in Z} z(b) = \top$. Then, there exists $b_1 \in B$, such that $z(b_1) = \top$ for every $z \in Z$. Then $\top = \bigwedge_{z\in Z}\bigvee_{b\in B} z(b) = \bigwedge_{z\in Z} z(\bigvee B) = g(\bigvee B)$. It follows that $g(\bigvee B) \geq \bigvee_{b\in B} g(b)$ and hence, together with the previously proved (converse) inequality, we have $g(\bigvee B) = \bigvee_{b\in B} g(b)$. Now, let $C \subseteq A$ be non-empty and finite, having $\bigwedge C \in A$. Then $g(\bigwedge C) = \bigwedge_{z\in Z} z(\bigwedge C) = \bigwedge_{z\in Z}\bigwedge_{c\in C} z(c) = \bigwedge_{c\in C}\bigwedge_{z\in Z} z(c) = \bigwedge_{c\in C} g(c)$. Finally, suppose that A has the greatest element $\bigwedge\emptyset \in A$. Then $g(\bigwedge\emptyset) = \bigwedge_{z\in Z} z(\bigwedge\emptyset) = \bigwedge_{z\in Z}\top = \top$. It follows that g is an element of $\langle A \to \mathbf{2}\rangle$, and, clearly, a lower bound of Z in $\langle A \to \mathbf{2}\rangle$. Let $l \in \langle A \to \mathbf{2}\rangle$ be a lower bound of Z. Then, for every $a \in A$ and every $z \in Z$ we have $l(a) \leq z(a)$, which gives $l(a) \leq \bigwedge_{z\in Z} z(a) = g(a)$ and, consequently, $l \leq g$. Therefore, g is a correctly defined infimum of Z in $\langle A \to \mathbf{2}\rangle$.

Regarding (iv), the mapping **True**, constantly equal to $\top$, obviously preserves all non-empty directed joins and all finite meets, so $\langle A \to \mathbf{2}\rangle$ also has the greatest element $\wedge\emptyset$. Note that **True** does not preserve the empty join, but it is not required.

Finally, to show (i), it remains to check that binary meets distribute over directed joins in $\langle A \to \mathbf{2}\rangle$. Let $x \in \langle A \to \mathbf{2}\rangle$ and $Y \subseteq \langle A \to \mathbf{2}\rangle$ be directed. Then $(x \wedge (\bigvee Y))(a) = x(a) \wedge (\bigvee Y)(a) = x(a) \wedge (\bigvee_{y\in Y} y(a)) = \bigvee_{y\in Y}(x(a) \wedge y(a)) = \bigvee_{y\in Y}((x \wedge y)(a)) = (\bigvee_{y\in Y}(x \wedge y))(a)$ for every $a \in A$, which implies $x \wedge (\bigvee Y) = \bigvee_{y\in Y}(x \wedge y)$. By the definition, $\langle A \to \mathbf{2}\rangle$ is a preframe.

□

It should be noted that there is a similar lemma also in our previous paper (4). We repeat it its slight reformulation because we need a more detailed formulation for the proof of the main theorem. Because of completeness and the reader's convenience, we also repeat yet another lemma with its brief proof. But first, let us make some additional denotations. Let A be a poset. We denote by $h_A : A \to \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ a mapping for which $(h_A(a))(x) = x(a)$ for every $x \in \langle A \to \mathbf{2}\rangle$. Further, for every $v \in \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ and $a \in A$ we put $h^*_A(v) = v \circ h_A$. The announced lemma now follows:

Lemma 2. Let A be a poset. Then $h_A : A \to \langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a morphism.

Proof. Let $B \subseteq A$ be non-empty and directed and suppose that there exists $\bigvee B \in A$. Let $x \in \langle A \to \mathbf{2}\rangle$. Then $h_A(\bigvee B)(x) = x(\bigvee B) = \bigvee_{b\in B} x(b) = \bigvee_{b\in B} h_A(b)(x) = (\bigvee_{b\in B} h_A(b))(x)$, which implies that $h_A(\bigvee B) = \bigvee_{b\in B} h_A(b)$. Let $C \subseteq A$ be non-empty, finite and assume that there exists $\bigwedge C \in A$. Let $x \in \langle A \to \mathbf{2}\rangle$. It follows $h_A(\bigwedge C)(x) = x(\bigwedge C) = \bigwedge_{c\in C} x(c) = \bigwedge_{c\in C} h_A(c)(x) = (\bigwedge_{c\in C} h_A(c))(x)$, which implies that $h_A(\bigwedge C) = \bigwedge_{c\in C} h_A(c)$. Finally, suppose that there exists the greatest element $\bigwedge\emptyset \in A$. It follows that $h_A(\bigwedge\emptyset)(x) = x(\bigwedge\emptyset) = \top$ for every morphism $x \in \langle A \to \mathbf{2}\rangle$, so $h_A(\bigwedge\emptyset) = \top$. Now, since h_A preserves all non-empty directed joins and all finite meets, including the empty meet, it follows that h_A is a morphism.

□

The next lemma is dual in some sense to the previous one:

Lemma 3. Let A be a poset. Then $h^*_A : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a morphism.

Proof. Let $v \in \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$. By the definition and the previous lemma, $h^*_A(v) = v \circ h_A$ is a morphism as a composition of two morphisms. So it is an element of $\langle A \to \mathbf{2}\rangle$, which means that h^*_A maps $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ into $\langle A \to \mathbf{2}\rangle$. It remains to show that it preserves the directed joins and the finite meets whenever they exist.

Let $V \subseteq \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ be non-empty and directed. The poset $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a preframe by Lemma 1., so the supremum $\bigvee V$ exists. Denote $V' = \{v \circ h_A |\, v \in V\}$. The set V' is a non-empty directed subset of $\langle A \to \mathbf{2}\rangle$ and again, by Lemma 1., it follows that $\bigvee V'$ exists. Moreover, $\bigvee V' = \bigvee_{v\in V} v \circ h_A = \bigvee_{v\in V} h^*_A(v)$. We will show that $h^*_A(\bigvee V)$ and $\bigvee V'$ are the same maps. For any $a \in A$ it holds $h^*_A(\bigvee V)(a) = ((\bigvee V) \circ h_A)(a) = (\bigvee V)(h_A(a))$. Using the statement (ii) of Lemma 1., we may continue: $(\bigvee V)(h_A(a)) = \bigvee_{v\in V} v(h_A(a)) = \bigvee_{v\in V}((v \circ h_A)(a)) = \bigvee_{w\in V'} w(a)$. Again, by the same statement (ii) of Lemma 1., we get

$\bigvee_{w\in V'} w(a) = (\bigvee V')(a)$. Hence, it follows $h_A^*(\bigvee V)(a) = (\bigvee V')(a)$ for every $a \in A$ and so $h_A^*(\bigvee V) = \bigvee V' = \bigvee_{v\in V} h_A^*(v)$.

Let $C \subseteq \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ be non-empty and finite. Since $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a preframe, the infimum $\bigwedge C$ exists. Let $C' = \{c \circ h_A |\, c \in C\}$. The set C' is a finite subset of $\langle A \to \mathbf{2}\rangle$ and again, by Lemma 1., it follows that $\bigwedge C'$ exists. We have $\bigwedge C' = \bigwedge_{c\in C} c \circ h_A = \bigwedge_{c\in C} h_A^*(c)$. We will prove that $h_A^*(\bigwedge C)$ and $\bigwedge C'$ are the same maps. Let $a \in A$. Then $h_A^*(\bigwedge C)(a) = ((\bigwedge C) \circ h_A)(a) = (\bigwedge C)(h_A(a))$. Using the statement (iii) of Lemma 1., we may continue as follows: $(\bigwedge C)(h_A(a)) = \bigwedge_{c\in C} c(h_A(a)) = \bigwedge_{c\in C}((c \circ h_A)(a)) = \bigwedge_{d\in C'} d(a)$. Using the statement (iii) of Lemma 1. once more, we have $\bigwedge_{d\in C'} d(a) = (\bigwedge C')(a)$. Therefore, $h_A^*(\bigwedge C)(a) = (\bigwedge C')(a)$ for every $a \in A$ and so $h_A^*(\bigwedge C) = \bigwedge C' = \bigwedge_{c\in C} h_A^*(c)$.

To complete the proof, as the last step we need to prove the preservation of the empty meet. However, the empty meet $\bigwedge \emptyset$, considered as an element of $\langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$, is a constant mapping, identically equal to $\top$ which takes arguments from $\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle$. Composed with h_A, we get a mapping defined on A, which is constantly equal to $\top$, that is, the top element **True** of $\langle A \to \mathbf{2}\rangle$.

□

We close the paper by the main theorem, which is the required analogue of the the second author's result $\tau^d \subseteq \tau^{ddd}$, proven in 2001 for the topological spaces. Among others, from the next theorem it follows that $h_{\langle A\to\mathbf{2}\rangle} : \langle A \to \mathbf{2}\rangle \to \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle$ is a monomorphism, which makes the analogy with the classical topological result from 2001 more obvious:

Theorem 4. Let A be a poset. Then $h_A^* : \langle\langle\langle A \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \mathbf{2}\rangle \to \langle A \to \mathbf{2}\rangle$ is a retraction.

Proof. We will show that $h_A^* \circ h_{\langle A\to\mathbf{2}\rangle} = id_{\langle A\to\mathbf{2}\rangle}$. Take $x \in \langle A \to \mathbf{2}\rangle$. Then $(h_A^* \circ h_{\langle A\to\mathbf{2}\rangle})(x) = h_A^*(h_{\langle A\to\mathbf{2}\rangle}(x)) = h_{\langle A\to\mathbf{2}\rangle}(x) \circ h_A$. Let $a \in A$. Then $((h_A^* \circ h_{\langle A\to\mathbf{2}\rangle})(x))(a) = (h_{\langle A\to\mathbf{2}\rangle}(x) \circ h_A)(a) = (h_{\langle A\to\mathbf{2}\rangle}(x))(h_A(a)) = (h_A(a))(x) = x(a)$. Now we have $(h_A^* \circ h_{\langle A\to\mathbf{2}\rangle})(x) = x$, which completes the proof.

□

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